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Geometrization of Linear Perturbation Theory for Diffeomorphism-Invariant Covariant Field Equations. I. The Notion of a Gauge-Invariant Variable

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Applying linear perturbation theory to the general-relativistic field equations, in a series of recent papers we have analyzed the gauge problem for an almost-Robertson–Walker universe. Mathematically, our analysis made use of a rather arbitrary choice of the background space-time geometry, and it turns out to possess the undesirable feature that the basic definitions and concepts are valid only for Einstein's gravity theory. The main purpose of this paper is to remedy all of the above deficiencies. Consequently, a new geometrical discussion of the notion of a gauge-invariant variable is presented with a view to demonstrating its usefulness in the context of an arbitrary diffeomorphism-invariant covariant field theory. Another welcome feature of this discussion is that, for linear perturbation theory, the proposed construction of gauge-invariant variables does not depend on the specific symmetry properties of the background "space-time" geometry chosen; in other words, it can be proven to hold for any possible choice of the background. In a companion paper, such an approach to the gauge problem will enable us to indicate in universal terms what geometrical objects are in fact essential if one is to obtain a fully satisfactory description of the equivalence classes of perturbations. A new example of the general structures, as compared with those already investigated for Einstein's gravity theory in the description of an almost-Robertson–Walker universe, is also given there. This example arises from consideration of the infinitesimal perturbation of the metric tensor itself (pure gravity) defined on a fixed background de Sitter space-time.

1. INTRODUCTION

Beginning from Einstein's theory of gravity, in a series of recent papers (Banach and Piekarski, 1996a–d) we have shown how to treat, in a fully

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covariant and gauge-invariant manner, the evolution of linear perturbations in homogeneous and isotropic cosmological models. To be more precise, we have developed there a systematic approach toward the problem of describing the equivalence classes of perturbations in a Robertson–Walker universe dominated by a general perfect fluid (Banach and Piekarski, 1996a–c) or consisting of massive collisionless particles (cold dark matter) (Banach and Piekarski, 1996d). Among other things, emphasis was placed on demonstrating that, for the aforementioned matter descriptions, a simple and unique characterization of cosmological perturbations can be obtained if one defines in a suitable way 17 or 18 "geometrically" independent, not identically vanishing gauge-invariant variables. These basic variables are important because they enable one to divide the infinitesimal perturbations into physically relevant equivalence classes: two infinitesimal perturbations $\delta_1 s_b$ and $\delta_2 s_b$ are said to be equivalent if there is a vector field v on the space-time manifold X such that $\delta_2 s_b$ differs from $\delta_1 s_b$ by the action of the Lie derivative $\mathcal{L}_v$ on the background solution s_b to nonlinear field equations. Another welcome feature of these variables is that any gauge-invariant quantity can be constructed directly from the basic variables through local (i.e., purely algebraic and differential) operations.

Given linear perturbation theory for Einstein's field equations, a systematic geometrical explanation of the notion of a gauge-invariant variable can be found in the paper by Banach and Piekarski (1996a). Although some important issues concerning the gauge problem were thereby resolved, the analysis of this paper nevertheless was not general in the following two senses: First, a proposal was made for defining the linearized perturbations as the equivalence classes of tangents to one-parameter families of exact solutions to the diffeomorphism-invariant, nonlinear field equations (Ehlers, 1973; Wald, 1984). However, this proposal relied heavily upon a particular diffeomorphism-invariant theory, namely Einstein's general theory of relativity. Second, since a definition of the notion of a gauge-invariant variable was given only for perturbations in Robertson–Walker universe models (Ryan and Shepley, 1975), it was not recognized that a completely general discussion (i.e., a construction of infinitely many gauge-invariant variables essentially independent of any concrete symmetry properties of the background space-time geometry chosen) almost always can be made. Consequently, several steps in the arguments were presented in an unnecessarily specific manner.

This is the first in a pair of papers (Banach and Piekarski, 1997), the overall objective of which is to remedy all of the above deficiencies. In particular, the basic object here is to gain more information concerning the systematic geometrical formulation of linear perturbation theory for the general case of covariant (diffeomorphism-invariant) field equations, rather than proceeding with those detailed calculations within any specific theory that are not universally valid or directly related to the general structure of a theory. As a consequence of this program, our construction of the gauge-invariant variable of order r will be proven to hold in an arbitrary diffeomorphism-invariant, covariant theory and for any possible choice of the background geometry. Interpreted from a slightly different point of view, this new construction thus gives a completely general proof, applicable to an arbitrary system of covariant field equations, of a result of Banach and Piekarski (1996a) for the case of general relativity. In a companion paper (Banach and Piekarski, 1997), such an approach to the gauge problem will enable us to indicate in universal terms what geometrical concepts, notions, and ideas are in fact essential if one is to obtain a fully satisfactory description of the equivalence classes of perturbations. Moreover, a new example of the general structures, as compared with those already presented (Banach and Piekarski, 1996a–d), is also given there. This example arises from consideration of the infinitesimal perturbation of the metric tensor itself (pure gravity) defined on a fixed background de Sitter space-time (Hawking and Ellis, 1973, p. 124).

The gauge problem for covariant field equations can be studied using the formalism of jet spaces (Vinogradov, 1984). Such a formalism is very general, enough so to obtain a natural geometrical definition of an arbitrary system of partial differential equations. Also, certain questions concerning the local symmetries on field configuration space of these equations are best illustrated by introducing the language of jet spaces. However, we shall be concerned here only with diffeomorphism-invariant (covariant) theories, i.e., the nonlinear field equations will be assumed to have a tensorial form in the precise sense explained in Section 2. Under these circumstances, as noted already by Lee and Wald (1990) and Iyer and Wald (1994), there is no loss of generality in employing a covariant derivative operator ∇ relative to an arbitrary, fixed, linear connection on the frame bundle of the manifold X of independent variables.[3] Our construction of gauge-invariant variables will make use of our choice of derivative operator on X. However, we will point out explicitly when the objects we define are independent of any such additional structure we have introduced. With the help of ∇, we then get rid of the jet spaces mentioned above by replacing them by a suitable hierarchy of tensor and vector bundles (Choquet-Bruhat *et al.*, 1989); the discussion can be made simpler thereby, yet without loss of generality of the conclusion. Next, we use this hierarchy of tensor and vector bundles in order to give an optimum geometrical explanation of the notion of a gauge-invariant variable. Nevertheless, it is perhaps important to stress that, for application to the present case (covariant theories), the formalism of jet spaces and that of

[3]Often the manifold X can be identified with the space-time manifold, even though this interpretation of X is not forced on us.

tensor and vector bundles are mathematically equivalent and it is possible to choose either one to suit the problem at hand; here we seek to study the concept of gauge invariance by means of the second formalism.

The layout of this paper is as follows. Section 2 introduces a number of tensor and vector bundles. This will serve to establish notation for the subsequent sections and to provide a concise characterization of covariant field equations. Section 3 presents the more relevant aspects of the "naive" gauge-dependent version of linear perturbation theory. Section 4 first defines the equivalence classes of perturbations and then considers the notion of a gauge-invariant variable of order r. [In a companion paper (Banach and Piekarski, 1997), we combine the results and constructions of this section to specify the conditions under which the equivalence classes of perturbations can be described in terms of a finite set of gauge-invariant variables. Within the framework of Einstein's gravity theory, these rather general conditions will be illustrated by considering the aforementioned case of a fixed background de Sitter space-time.] Section 5 gives an interpretation of how the notion of a gauge-invariant variable relates to the properties of the equations governing linearized perturbations. Section 6 is for discussion and conclusion. Finally, the auxiliary technical material is included as an Appendix.

2. PRELIMINARIES

2.1. Some Useful Tensor and Vector Bundles

The objective here in Section 2.1 is to define a suitable hierarchy of tensor and vector bundles; this will, among other things, serve as a convenient starting point for any systematic discussion of the general structure of diffeomorphism-invariant, covariant field equations (see Section 2.2). The basic idea is in fact quite simple. In our approach to covariant field theories, it is natural to introduce a finite set $\tilde{s}$ of tensor fields s^A ($A = 1, 2, \ldots, n$) as fundamental objects, to take s^A to be a cross section of the appropriate tensor bundle S^A (Choquet-Bruhat *et al.*, 1989; Dieudonné, 1972), and to assume that $\tilde{s}$ or alternatively[4]

$$s := \bigoplus_{A=1}^{n} s^A \tag{2.1}$$

is a classical solution of the nonlinear system of covariant field equations [see equations (2.23) in Section 2.2]. Clearly, from this point of view, s can

[4]Under the canonical injections (Greub, 1975, pp. 56–60), the set $\tilde{s}$ of tensor fields s^A ($A = 1, 2, \ldots, n$) is an element of S, the (external) direct sum of S^A. Consequently, given the definitions (2.1) and (2.2) of s and S, it may be convenient not to distinguish between $\tilde{s}$ and s, but to regard them as the same geometrical object.

be identified with, or represented by, a cross section of the vector bundle S which is a direct sum of S^A:

$$S := \bigoplus_{A=1}^{n} S^A \tag{2.2}$$

We further specify S^A (and hence S) as follows. Let X be an N-dimensional manifold; we refer to this manifold as the "space-time" manifold or, more precisely, the manifold of independent variables. Then, by using the same terminology as in Dieudonné (1972, p. 119), we conclude that there should be no essential loss of generality in assuming that S^A is the tensor bundle of type (r_A, R_A) over X (r_A is the "contravariant" index and R_A is the "covariant" index); for simplicity we shall assume this is the case. Let S_x^A be a fiber of S^A over $x \in X$. This fiber consists of such tensors at $x \in X$ of covariant valence R_A and contravariant valence r_A that S_x^A can be endowed with a structure of a vector space of $N^{r_A+R_A}$ dimensions ($\dim S_x^A = N^{r_A+R_A}$). After these preparations, we easily define S^A in terms of S_x^A:

$$S^A := \bigcup_{x \in X} S_x^A \tag{2.3}$$

Given a finite set of tensor bundles S^A ($A = 1, 2, \ldots, n$), we can construct new bundles by a variety of operations. One such operation is the direct sum of S^A ($A = 1, 2, \ldots, n$):

$$S := \bigoplus_{A=1}^{n} S^A := \bigcup_{x \in X} S_x \tag{2.4}$$

This operation is accomplished by taking S to be the disjoint union of the vector spaces

$$S_x := \bigoplus_{A=1}^{n} S_x^A \tag{2.5}$$

as x runs through X. It follows from equations (2.4) and (2.5) that S carries a natural structure of a vector bundle. The importance of S will appear clearly when we come to the study of diffeomorphism-invariant (i.e., covariant) field equations. Here we only mention that any solution of these equations can be identified with the mapping s: $X \to S$ which is a cross section of S:

$$X \ni x \mapsto s(x) \in S_x \tag{2.6}$$

Such an interpretation of s again yields [see equation (2.1)]

$$s(x) = \bigoplus_{A=1}^{n} s^A(x) \tag{2.7}$$

where $s^A(x)$ stands for some element of S_x^A.

Taking the dual is another operation for creating new bundles from old. Let S_x^{A*} and S_x^* be a pair of vector spaces dual to S_x^A and S_x, respectively:

$$S_x^* := \bigoplus_{A=1}^{n} S_x^{A*} \tag{2.8}$$

For any A, put

$$S^{A*} := \bigcup_{x \in X} S_x^{A*} \tag{2.9}$$

so that S^{A*} is a tensor bundle whose fiber S_x^{A*} over $x \in X$ consists of tensors at $x \in X$ of contravariant valence R_A and covariant valence r_A. Then we can define a new vector bundle over X which is called the dual of S and is denoted by S^*:

$$S^* := \bigoplus_{A=1}^{n} S^{A*} := \bigcup_{x \in X} S_x^* \tag{2.10}$$

In this paper, the idea of introducing the vector bundles dual to the original ones lends itself to interpretation in two complementary senses: first, as a way to understand the general structure of the equations governing linearized perturbations (see Section 3), and second, as a way to discuss the concept of gauge invariance (see Section 4).

From the considerations in Section 2.2 below we infer that it helps in analyzing the covariant field equations to consider the tensor bundles S_p^A defined as follows. For each integer $p \geq 0$ and any A, denote by $S_{p,x}^A$ the set of tensors at $x \in X$ of type $(r_A, R_A + p)$. To see the structure of this set in a better light, we can alternatively characterize $S_{p,x}^A$ by saying that if ∇ represents the covariant derivative relative to any linear connection on X (Dieudonné, 1972, p. 333) and $\nabla^p s^A := \nabla(\nabla^{p-1} s^A)$ describes the effect of a multiple application of ∇ to s^A (with s^A being an arbitrary C^p cross section of S^A), then the image of $x \in X$ under $\nabla^p s^A$ (i.e., the value of $\nabla^p s^A$ at $x \in X$) is an element of $S_{p,x}^A$. Now, given the above definition of $S_{p,x}^A$, let S_p^A be the disjoint union of the sets $S_{p,x}^A$ as x runs through X; thus

$$S_p^A := \bigcup_{x \in X} S_{p,x}^A \tag{2.11}$$

For essentially obvious reasons, S_p^A carries a unique tensor bundle structure with base X and canonical projection π_p^A, and S_p^A equals S^A when $p = 0$ $(\pi^A := \pi_0^A)$.

Next, let $S_{p,x}^{A*}$ be a vector space dual to $S_{p,x}^A$. This yields the dual of S_p^A as given by

$$S_p^{A*} := \bigcup_{x \in X} S_{p,x}^{A*} \tag{2.12}$$

Using $S^A_{p,x}$ and $S^{A*}_{p,x}$, we can also define the following vector spaces:

$$S_{p,x} := \bigoplus_{A=1}^{n} S^A_{p,x} \tag{2.13a}$$

$$S^*_{p,x} := \bigoplus_{A=1}^{n} S^{A*}_{p,x} \tag{2.13b}$$

$$S_{(r),x} := \bigoplus_{p=0}^{r} S_{p,x} \tag{2.14a}$$

$$S^*_{(r),x} := \bigoplus_{p=0}^{r} S^*_{p,x} \tag{2.14b}$$

In addition to S and S^*, these definitions permit us to introduce four new vector bundles over X:

$$S_p := \bigcup_{x\in X} S_{p,x} = \bigoplus_{A=1}^{n} S^A_p \tag{2.15a}$$

$$S^*_p := \bigcup_{x\in X} S^*_{p,x} = \bigoplus_{A=1}^{n} S^{A*}_p \tag{2.15b}$$

$$S_{(r)} := \bigcup_{x\in X} S_{(r),x} = \bigoplus_{p=0}^{r} S_p \tag{2.16a}$$

$$S^*_{(r)} := \bigcup_{x\in X} S^*_{(r),x} = \bigoplus_{p=0}^{r} S^*_p \tag{2.16b}$$

Here and in what follows, the choice of an integer $r \geq 0$ in the definitions (2.14) and (2.16) is left arbitrary. A particularly important cross section of S_p is determined by

$$X \ni x \mapsto \bigoplus_{A=1}^{n} (\nabla^p s^A)_x \in S_{p,x} \tag{2.17}$$

where, as usual, the symbol $(\nabla^p s^A)_x$ denotes the value of $\nabla^p s^A$ at $x \in X$. Also, elementary inspection shows that

$$X \ni x \mapsto \bigoplus_{p=0}^{r} \left[\bigoplus_{A=1}^{n} (\nabla^p s^A)_x \right] \in S_{(r),x} \tag{2.18}$$

is a cross section of $S_{(r)}$. Consequently, the vector bundles S_p and $S_{(r)}$ are very useful and appear naturally in any discussion where one considers the first-, second-, and higher order covariant derivatives of s. In Sections 3 and 4, a study of further properties concerned with the linearized field equations and the gauge problem will show the usefulness and naturalness of dual vector bundles S_p^* and $S_{(r)}^*$ as well.

2.2. Diffeomorphism-Invariant (Covariant) Field Equations

To proceed further, we must be somewhat more specific about the choice of an invariant type of differentiation of tensor fields. In each particular case, especially for metric theories of gravity (Iyer and Wald, 1994) or some unified field-theoretic treatments of gravitation (Slawianowski, 1994, 1995), there will always be a preferred linear connection on X, so that one can employ a covariant derivative based on this connection. Usually, such a covariant derivative is denoted by ∇. Since, however, the overall objective here is to formulate linear perturbation theory for the general case of diffeomorphism-invariant, covariant field equations, this paper will instead adopt a more universal attitude. *Thus it will be assumed henceforth that* ∇ *is the covariant derivative relative to an arbitrary, fixed, linear connection on* X.

Another, equally legitimate, choice which one might have made in many situations would be to still use a preferred linear connection on X. However, as follows from the discussion in Sections 3 and 5, our choice, albeit at first sight unnecessary, has the distinct practical advantage of considerably simplifying the derivation of linear perturbation equations. In this context, it should be noted that what fixed linear connection on X one uses is really unimportant: everything will make perfectly good sense provided only that the value of a difference between the "preferred" and fixed covariant derivatives of $\nabla^p s^A$ ($p \geq 0$) at $x \in X$ can be "constructed" locally out of the quantities $(\nabla^k s^B)_x$, where $k = 0, 1, \ldots, p + 1$ and $B = 1, 2, \ldots, n$ (here, of course, such a construction is permitted to depend explicitly on $x \in X$). As an illustration, in discussing the Cauchy problem and/or linear perturbation theory for Einstein's field equations and their modifications such as, e.g., the Brans–Dicke and Hellings–Nordtvedt equations (Brans and Dicke, 1961; Hellings and Nordtvedt, 1973), it is the custom to introduce a fixed *"background" metric* g_b as well as the physical metric g and then to consider a *tensor field* on X which is the difference between the connection defined by g and the connection defined by g_b. With the help of this tensor object, one can then prove that the original equations of metric theories of gravity can be expressed in terms of the covariant derivatives of various tensor fields with respect to g_b. For further details concerning these issues, see the comprehensive analysis by Iyer and Wald (1994, p. 847).

In connection with our treatment of diffeomorphism-invariant, covariant field theories, another useful digression is also needed. Adopting the standard convention of setting $\nabla^p s^A = s^A$ when $p = 0$, for each integer $p \geq 0$ we define the action of ∇^p on $s := \oplus_{A=1}^n s^A$ as follows:

$$\nabla^p s := \bigoplus_{A=1}^{n} \nabla^p s^A \quad (\Rightarrow \nabla s := \nabla^1 s) \tag{2.19}$$

Further, we specify the linear operator D^r by explicitly describing its effect on the C^r cross sections $x \mapsto s(x)$ of S:

$$D^r s := \bigoplus_{p=0}^{r} \nabla^p s \tag{2.20}$$

Here r is an arbitrary integer ($r \geq 0$). Then, after denoting by $(\nabla^p s)_x$ and $(D^r s)_x$ the values of $\nabla^p s$ and $D^r s$ at $x \in X$, the mappings $x \mapsto (\nabla^p s)_x$ and $x \mapsto (D^r s)_x$ give an interpretation of $\nabla^p s$ and $D^r s$ as cross sections of S_p and $S_{(r)}$, respectively.

Now, with this preparation behind us, let it be supposed that the mapping $s: X \to S$, which is a cross section of the vector bundle S, satisfies the following system of field equations:

$$H^I(x, s, \nabla s, \ldots, \nabla^q s) = 0, \qquad I = 1, 2, \ldots, m \tag{2.21}$$

Here, as well as in our considerations below, the value of H^I at a point $x \in X$ depends only on the values of s and its covariant derivatives $\nabla^p s$ up to order q evaluated at a point $x \in X$. However, to make the resulting formulas shorter, in equations (2.21) the dependence of $s, \nabla s, \ldots, \nabla^q s$ upon x is not shown explicitly:

$$H^I[x, s(x), (\nabla s)_x, \ldots, (\nabla^q s)_x] = H^I(x, s, \nabla s, \ldots, \nabla^q s) \tag{2.22}$$

Clearly, on employing the definition (2.20) of $D^r s$, we may always write equations (2.21) in the abbreviated form

$$H^I(x, D^q s) = 0, \qquad I = 1, 2, \ldots, m \tag{2.23}$$

From now on it will be postulated that, for each C^q cross section $x \mapsto s(x)$ of S, the objects $H^I(\cdot, s, \nabla s, \ldots, \nabla^q s)$ and/or $H^I(\cdot, D^q s)$ are tensor fields of various types on X (Dieudonné, 1972, p. 119). This postulate can be viewed as expressing the invariance of the component form of field equations under general coordinate transformations. Consequently, we shall be concerned here only with diffeomorphism-invariant theories, i.e., the objects H^I, $I = 1, 2, \ldots, m$, will be assumed to be diffeomorphism-covariant in the sense that

$$\sigma * H^I(\cdot, D^q s) = H^I[\cdot, D^q(\sigma * s)] \tag{2.24}$$

where $\sigma*$ is the action induced on the fields by a diffeomorphism $\sigma: X \to X$ (Iyer and Wald, 1994, p. 847). From this viewpoint, equations (2.21) or (2.23) may be interpreted as covariant field equations. For obvious reasons, we restrict our attention to situations where these equations form a determinate system of partial differential equations (i.e., this system is neither overdetermined nor underdetermined).

Characterizing the properties of equations (2.21) and/or (2.23) still further, we also note that the statement that the objects H^I, $I = 1, 2, \ldots, m$, depend for each $x \in X$ only on $s(x)$ and the first q covariant derivatives $\nabla^p s$ of s evaluated at $x \in X$ is completely independent of the choice of derivative operator on X. Because of this, it seems reasonable to refer to equations (2.21) or (2.23) as the covariant field equations of order q. Naturally, if we are primarily interested in some modifications of a classical theory of gravity, then the original collection of tensor fields, namely $\{s^A; A = 1, 2, \ldots, n\}$, consists of a (Lorentz signature) metric g ($s^1 := g$) and other tensor fields s^A, $A \neq 1$, on X. However, no such identification of one of the fields with g is necessary in the general case, and our constructions below are valid even for those diffeomorphism-invariant theories in which the aforementioned collection of fields does not originally contain a metric g (Slawianowski, 1994, 1995).

Finally, to illustrate the use of the vector bundle $S_{(q)}$ with base X and canonical projection $\pi_{(q)}$, we now give a geometrical definition of equations (2.23). To this end, consider first the set $\Gamma_{(q)}$ which consists of all cross sections of $S_{(q)}$. Letting s_p^A denote the cross section of S_p^A and putting

$$s_p := \bigoplus_{A=1}^{n} s_p^A \tag{2.25}$$

we find from

$$s_{(q)} := \bigoplus_{p=0}^{q} s_p \tag{2.26}$$

that the mapping

$$X \ni x \mapsto s_{(q)}(x) \in S_{(q)x} \tag{2.27}$$

is an element of $\Gamma_{(q)}$. Since the vector bundle $S_{(q)}$ is a manifold (i.e., a finite-dimensional space where some neighborhood of each point admits a coordinate system), the set $\Gamma_{(q)}$ serves to promote the interpretation of $u \in S_{(q)}$ as a pair $(x, s_{(q)}(x))$ in which $x := \pi_{(q)}u$ and $s_{(q)}$ is such a cross section of $S_{(q)}$ that $s_{(q)}(x) = u$; the choice of $s_{(q)} \in \Gamma_{(q)}$ depends of course on the choice of $u \in S_{(q)}$, but nevertheless is not unique. Under this interpretation, and after abbreviating $(x, s_{(q)}(x))$ as $(x, s_{(q)})$, we can uniquely define the subset or submanifold $\mathcal{Y}$ of $S_{(q)}$ by saying that u belongs to $\mathcal{Y}$ if and only if the corresponding pair $(x, s_{(q)})$ satisfies the conditions of the form

$$H^I(x, s_{(q)}) = 0, \qquad I = 1, 2, \ldots, m \tag{2.28}$$

Here $s_{(q)}$ plays the role previously played by $D^q s$ in equations (2.23). The above conditions define $\mathcal{Y}$ by an "algebraic" system of equations, *and it is possible and natural to identify the covariant field equations with* $\mathcal{Y}$. Writing Γ for the vector space of cross sections of S, we then see that $s \in \Gamma$ is a solution of equations (2.23) if and only if the set

$$\mathcal{Y}_s := \{(x, D^q s): x \in X\} \tag{2.29}$$

is a subset of $\mathcal{Y}$.

These definitions of $\mathcal{Y}$ and $\mathcal{Y}_s$ are analogous to those proposed by Vinogradov (1984) in the context of an arbitrary system of partial differential equations, the vector bundles $S_{(r)}$ being replaced by the jet spaces $J^r(\pi)$, and one might ask why we should spend our time on $S_{(r)}$ when a formalism of jet spaces solves the problem completely. The answer is that the practical feasibility of getting a meaningful geometric definition of equations (2.23) depends very much on the complexity of the formalism involved. For covariant field equations, it happens that the method based on $S_{(r)}$ is much simpler than that based on $J^r(\pi)$. The same conclusion holds if we decide to discuss in greater detail the structural features of a linear approximation to these generally nonlinear field equations.

3. LINEAR PERTURBATION EQUATIONS

3.1. Discussion of the Special Case When H^I ($I = 1, 2, \ldots, m$) Are Scalar Fields on X

In an exact description, the full nonlinear system of equations, which consists of equations (2.21) or (2.23), would become a complicated set of equations for the determination of s. Clearly, solving this system is not simple, but it turns out that we are often interested only in solutions s close to a given "background" solution s_b and for those solutions another device worth noting is that of using perturbation theory to obtain a linearized form of equations (2.21) or (2.23). The basic assumptions of this theory, which seem necessary in order to give a clear idea of what the perturbation method is to be (Ehlers, 1973; Banach and Piekarski, 1994a, b), may be formulated as follows: Consider an open interval $\mathcal{U} := (-d, d)$ of R, $d > 0$. Assume further that for each $\epsilon \in \mathcal{U}$ there exists a classical solution $s(\epsilon, x)$ to equations (2.21) or (2.23):

$$s(\epsilon, x) = \bigoplus_{A=1}^{n} s^A(\epsilon, x) \tag{3.1}$$

In view of our definitions in Section 2, we may interpret equation (3.1) as asserting that $s^A(\epsilon, x)$ is an element of S_x^A and $s(\epsilon, x)$ is an element of S_x; this interpretation holds for each $x \in X$ and each $\epsilon \in \mathcal{U}$. The set of classical solutions defines a function space, and a *one-parameter family* of exact solutions given by $\{s(\epsilon, \cdot); \epsilon \in \mathcal{U}\}$ may be thought of as a curve in the function space passing through the "point" $s_b := s(\epsilon, \cdot)_{|\epsilon=0}$, which we call the *background solution*. Putting it differently, we may say that the parameter $\epsilon \in \mathcal{U}$ measures the size of the perturbation in the sense that the fields s^A ($A = 1, 2, \ldots, n$) depend continuously on ϵ and the object

$$s_b(x) := s(0, x) = \bigoplus_{A=1}^{n} s^A(0, x) \tag{3.2}$$

is a *known* solution of nonlinear field equations (Wald, 1984). Now, abbreviating the value of $s^A(\epsilon, x)$ at $\epsilon = 0$ as $s_b^A(x)$ [$s_b^A(x) := s^A(0, x)$], we find from equation (3.2) that

$$s_b(x) = \bigoplus_{A=1}^{n} s_b^A(x) \tag{3.3}$$

If the fields s^A ($A = 1, 2, \ldots, n$) depend differentiably on ϵ, it will also be possible to define the *infinitesimal perturbation* of s_b^A as follows:

$$(s^A)' := \left(\frac{\partial s^A}{\partial \epsilon}\right)_{\epsilon=0} \tag{3.4}$$

Similarly, we can define the infinitesimal perturbation of s_b:

$$s' := \left(\frac{\partial s}{\partial \epsilon}\right)_{\epsilon=0} = \bigoplus_{A=1}^{n} (s^A)' \tag{3.5}$$

In the Introduction, we denoted this perturbation by $\delta_1 s_b$ or $\delta_2 s_b$. Note that $(s^A)'(x)$ and $s'(x)$, the values of $(s^A)'$ and s' at $x \in X$, are the tangents to the curves $c_x^A(\epsilon) = s^A(\epsilon, x)$ and $c_x(\epsilon) = s(\epsilon, x)$ (with x fixed) in S_x^A and S_x at $\epsilon = 0$, so $(s^A)'(x)$ and $s'(x)$ may naturally be viewed as vectors in the tangent spaces to S_x^A and S_x at the "points" $s_b^A(x)$ and $s_b(x)$. However, due to the vector space structure of both S_x^A and S_x, it is possible to identify the tangent space at $s_b^A(x)$ or $s_b(x)$ with S_b^A and S_x itself. Under this identification, equations (3.4) and (3.5) clearly show that the mappings $x \mapsto (s^A)'(x)$ and $x \mapsto s'(x)$ are cross sections of S^A and S, respectively.

Here it should perhaps be stressed that one could have approached the problem of defining the infinitesimal perturbation s' of s_b in a number of other ways. Thus, for example, one might have sought to regard $\{s(\epsilon, \cdot); \epsilon \in \mathcal{U}\}$ not as a one-parameter family of exact solutions to equations (2.21), but rather as an arbitrary C^1 curve in the function space passing through the background solution $s_b(\cdot)$. In this way of thinking, the infinitesimal perturbation s' of s_b defined by equation (3.5) is an arbitrary cross section of S. We shall repeatedly make use of this extended definition of s' in what follows (see especially our discussion at the end of Section 4.1, where we consider the equivalence class $[s']$ of $s' \in \Gamma$).

For convenience, and before deriving the linear field equations for s', we introduce the following useful notation for the value of $\nabla^p s(\epsilon, \cdot)$ at $x \in X$:

$$w^p(\epsilon, x) := (\nabla^p s(\epsilon, \cdot))_x \in S_{p,x}, \qquad p = 0, 1, \ldots, \infty \tag{3.6}$$

In using this notation, it is understood that $s(\epsilon, \cdot)$ satisfies equations (2.21) and/or (2.23) if ϵ belongs to $\mathcal{U}$. Consequently, using the definition (3.6), we may rewrite these equations as

$$H^I[x, w^0(\epsilon, x), w^1(\epsilon, x), \ldots, w^q(\epsilon, x)] = 0 \tag{3.7}$$

with an integer I ranging from 1 to m. Under appropriate assumptions of uniformity, we have

$$\left(\frac{\partial w^p}{\partial \epsilon}\right)_{\epsilon=0} = \nabla^p s' = \bigoplus_{A=1}^{n} \nabla^p (s^A)' \tag{3.8}$$

because the derivative operator ∇ *on* X *does not depend on* ϵ. Hence the linear field equations for s' are most easily obtained by first differentiating H^I with respect to ϵ at $\epsilon = 0$ and then exploiting the obvious relations

$$\left(\frac{\partial H^I}{\partial \epsilon}\right)_{\epsilon=0} = 0, \qquad I = 1, 2, \ldots, m \tag{3.9}$$

As shown below, equations (3.9) indeed are linear equations for s', i.e., they can be expressed in the form

$$Lop^I(s') = 0, \qquad I = 1, 2, \ldots, m \tag{3.10}$$

where Lop^I is a linear differential "space-time" operator acting on s'. If we can solve equations (3.10) for s', then $s_b(x) + \epsilon s'(x)$ should yield a good approximation to $s(\epsilon, x)$ near $\epsilon = 0$, and issues of practical interest thus can be investigated.

In order to explicitly construct Lop^I, consider first the special case of equations (3.7) in which H^I ($I = 1, 2, \ldots, m$) are usual scalar functions of $x, w^0, w^1, \ldots, w^q$ of class C^1 with respect to w^r ($r = 0, 1, \ldots, q$). Since, then, for each possible choice of (I, x) and w^r ($r \neq p$), the object $H^I(x, w^0, w^1, \ldots, w^{p-1}, \bullet, w^{p+1}, \ldots, w^q)$ is a differentiable mapping of $S_{p,x}$ into $\mathbb{R}$, we are naturally led to define the "derivative of H^I with respect to $\overline{w}^p \in S_{p,x}$ at $\overline{w}^p = w^p$" to be a linear form on $S_{p,x}$, i.e., an element of the vector space $S^*_{p,x}$ dual to $S_{p,x}$. *More precise specifications concerning this derivative, which* *for brevity we denote as dH^l/dw^p, occur in the Appendix.* Here we only mention that $dH^l/dw^p \in S^*_{p,x}$ depends on $w^0, w^1, \ldots, w^q$ and hence on $\epsilon \in \mathcal{U}$ through $w^r = w^r(\epsilon, x)$, $r = 0, 1, \ldots, q$. It is possible to express this fact by writing

$$\frac{dH^l}{dw^p} = \mathcal{H}^l_p[x, w^0(\epsilon, x), w^1(\epsilon, x), \ldots, w^q(\epsilon, x)]$$
$$= \mathcal{H}^l_p(x, w^0, w^1, \ldots, w^q) \tag{3.11}$$

Now, using the identity $w^r(\epsilon, x)_{\epsilon=0} = (\nabla^r s_b)_x$ [of interest if only the standard assumptions of uniformity mentioned above are valid; see the text directly before equation (3.8)] and subsequently letting $H^l_p \in S^*_{p,x}$ be the value of $\mathcal{H}^l_p \in S^*_{p,x}$ at $\epsilon = 0$, i.e., setting

$$H^l_p(x) := (\mathcal{H}^l_p)_{\epsilon=0}$$
$$= \mathcal{H}^l_p[x, (\nabla^0 s_b)_x, (\nabla^1 s_b)_x, \ldots, (\nabla^q s_b)_x]$$
$$= \mathcal{H}^l_p(x, s_b, \nabla s_b, \ldots, \nabla^q s_b) \tag{3.12}$$

we can explicitly calculate $(\partial H^l/\partial\epsilon)_{\epsilon=0}$ (and hence Lop^l) by contracting

$$H^l_{(q)}(x) := \bigoplus_{p=0}^{q} H^l_p(x) \in S^*_{(q),x} \tag{3.13}$$

with

$$(D^q s')_x := \bigoplus_{p=0}^{q} (\nabla^p s')_x \in S_{(q),x} \tag{3.14}$$

More precisely, we arrive at

$$\left(\frac{\partial H^l}{\partial \epsilon}\right)_{\epsilon=0} = \langle H^l_{(q)}, D^q s' \rangle = \sum_{p=0}^{q} H^l_p \odot \nabla^p s' \tag{3.15}$$

where the bilinear function $\langle \cdot, \cdot \rangle$ is a *natural pairing* of $S^*_{(q),x}$ and $S_{(q),x}$ into R (Bishop and Goldberg, 1968, p. 77) and where the symbol $\odot$ indicates that $H^l_p \odot \nabla^p s'$ is a value of $H^l_p \in S^*_{p,x}$ on $\nabla^p s' \in S_{p,x}$ (i.e., a contraction of H^l_p with $\nabla^p s'$).

In summary, from equations (3.9) and (3.15) we get

$$\langle H^l_{(q)}, D^q s' \rangle = 0, \qquad l = 1, 2, \ldots, m \tag{3.16a}$$

Equivalently, we obtain

$$\sum_{p=0}^{q} H^l_p \odot \nabla^p s' = 0, \qquad l = 1, 2, \ldots, m \tag{3.16b}$$

Each of the results (3.16a) and (3.16b) is the desired system of linear differential equations for the determination of s'. The virtues of the description based on equations (3.16a) come to the fore when it is convenient to reduce linear perturbation theory to the simplest possible form—for instance, when discussing the gauge problem. The description based on equations (3.16b) has complementary advantages when it is useful to see the role of ∇—for example, when illustrating the economy of the covariant approach vis-à-vis the technique of jet spaces (Vinogradov, 1984). As noted already (see Section 2.2), for theories in which one of the fields s^A is the metric g ($s^1 = g$) it seems natural to think of ∇ as being the covariant derivative based on the background solution s_b ($s_b^1 = g_b$). However, this interpretation of ∇ is not forced on us, and for a broad class of nonlinear and linearized field theories we can also choose ∇ to be the covariant derivative with respect to an *arbitrary, fixed, linear connection on X*. Such is indeed the case, because the operators Lop^I appearing on the left-hand side of equations (3.10) are independent of the choice of ∇. Of course, after the linearized field equations have been derived, it proves helpful to regard the mappings [see equations (3.16a) and (3.16b)]

$$X \ni x \mapsto H_p^I(x) \in S_{p,x}^* \tag{3.17}$$

and

$$X \ni x \mapsto H_{(q)}^I(x) \in S_{(q),x}^* \tag{3.18}$$

as cross sections of S_p^* and $S_{(q)}^*$, respectively. Thus, if $H^I(x, \cdot)$, $I = 1, 2, \ldots, m$, are real-valued functions on $S_{(q),x}$, one sees why it is that the vector bundles dual to S_p and $S_{(q)}$ play such a large part in linear perturbation theory. These vector bundles appear to be important also in the general situation described below.

3.2. Discussion of the General Case When H^I ($I = 1, 2, \ldots, m$) Are Arbitrary Tensor Fields on X

The analysis so far presented in this section has been based on the assumption that, for each I, we may take H^I to be a mapping of $S_{(q)}$ into R. In the general case considered here, the value of H^I at $u \in S_{(q)}$ is of course a tensor at $x := \pi_{(q)}u$. Let V_x^I be a vector space to which this tensor belongs, and denote by V^I the disjoint union of the vector spaces V_x^I as x runs through X and by π_I the mapping $V^I \to X$ which sends each element of V_x^I to x. Then V^I is canonically endowed with a vector bundle structure over X. These definitions may be combined with each other and with the preceding ones to give an interpretation of H^I as the mapping of $S_{(q)}$ into V^I with the following property: for each $u \in S_{(q)}$, the image of u under $\pi_{(q)}$ equals the image of $H^I(u)$ under π_I.

Now, for each $x \in X$, let $\{e_I^K(x)\}$ be a basis of V_x^I. Then the sections $x \mapsto e_I^K(x)$ form a frame $\{e_I^K\}$ of V^I over X, so that the mapping H^I: $S_{(q)} \to V^I$ defined above can be written as

$$H^I = \sum_K H_K^I e_I^K \tag{3.19}$$

where H_K^I are real-valued functions on $S_{(q)}$ and where K is an integer which ranges from 1 to $m_I := \dim V_x^I$ (as usual, it will be assumed that the dimension of V_x^I does not depend on $x \in X$). Instead of considering equations (2.21), we may thus *equivalently* consider the *scalar equations*

$$H_K^I(x, s, \nabla s, \ldots, \nabla^q s) = 0 \tag{3.20}$$

with $I = 1, 2, \ldots, m$ and $K = 1, 2, \ldots, m_I$. Since it comes to the same thing to require that $s \in \Gamma$ satisfies equations (2.21) or equations (3.20), all the definitions and all the results of Section 3.1 remain valid, *mutatis mutandis*, when we replace H^I by H_K^I in the statements and proofs. For example, setting

$$H_{Kp}^I := \left(\frac{dH_K^I}{dw^p}\right)_{\epsilon=0} \in S_{p,x}^* \tag{3.21}$$

and

$$H_{K(q)}^I := \bigoplus_{p=0}^{q} H_{Kp}^I \in S_{(q),x}^* \tag{3.22}$$

we find that $s' \in \Gamma$ is constrained to satisfy the following system of linear perturbation equations:

$$\langle H_{K(q)}^I, D^q s' \rangle = 0 \tag{3.23a}$$

$$I = 1, 2, \ldots, m, \qquad K = 1, 2, \ldots, m_I \tag{3.23b}$$

Hence, using the definitions (3.14) and (3.22), we get

$$\sum_{p=0}^{q} H_{Kp}^I \odot \nabla^p s' = 0 \qquad (I = 1, 2, \ldots, m; K = 1, 2, \ldots, m_I) \tag{3.24}$$

establishing thereby the analog of equations (3.16b).

If we bear in mind the definition of H_K^I by means of equations (3.19), we can of course seek by these means to bring everything back to the definitions of Section 3.1, but it is essential to verify that in this way we obtain notions which are *intrinsic* to linear perturbation theory, that is, which do not depend on the choice of $\{e_I^K\}$. Now, once we have introduced the frame-dependent objects $H^I_{Kp}(x) \in S^*_{p,x}$ and $H^I_{K(q)}(x) \in S^*_{(q),x}$, the only notions which appear to be intrinsic are as follows. First, consider the linear mappings $H^I_p(x)$: $S_{p,x} \to V^I_x$ and $H^I_{(q)}(x)$: $S_{(q),x} \to V^I_x$ given by

$$H^I_p(x) := \sum_K H^I_{Kp}(x) e^K_I(x) \tag{3.25}$$

and

$$H^I_{(q)}(x) := \sum_K H^I_{K(q)}(x) e^K_I(x) \tag{3.26}$$

Then, clearly, after denoting by $H^I_p(x) \odot s_p(x)$ the image of $s_p(x) \in S_{p,x}$ under $H^I_p(x)$ and by $\langle H^I_{(q)}(x), s_{(q)}(x)\rangle$ the image of $s_{(q)}(x) \in S_{(q),x}$ under $H^I_{(q)}(x)$, we have

$$H^I_p(x) \odot s_p(x) := \sum_K [H^I_{Kp}(x) \odot s_p(x)] e^K_I(x) \tag{3.27}$$

and

$$\langle H^I_{(q)}(x), s_{(q)}(x)\rangle := \sum_K \langle H^I_{K(q)}(x), s_{(q)}(x)\rangle e^K_I(x) \tag{3.28}$$

Elementary inspection shows that we cannot define the mappings $H^I_{Kp}(x)$: $S_{p,x} \to \mathrm{R}$ and $H^I_{K(q)}(x)$: $S_{(q),x} \to \mathrm{R}$ intrinsically; but it is perfectly possible to use $H^I_p(x)$ and $H^I_{(q)}(x)$ in place of them: for it can be verified that, although the forms of $H^I_{Kp}(x)$ and $H^I_{K(q)}(x)$ depend on the choice of $\{e^K_I(x)\}$, nevertheless the mappings (3.25) and (3.26) investigated above do not depend on the particular $\{e^K_I\}$ chosen. With all these preparatory statements behind us, the key point is in fact quite simple. Combining equations (3.24) and (3.27) yields

$$\sum_{p=0}^{q} H^I_p \odot \nabla^p s' = 0, \qquad I = 1, 2, \ldots, m \tag{3.29}$$

this being the analog of equations (3.16b). Next, by applying the definition (3.28) to equations (3.23) we also find that

$$\langle H^I_{(q)}, D^q s'\rangle = 0, \qquad I = 1, 2, \ldots, m \tag{3.30}$$

The analogy to equations (3.16a) is then immediate. In connection with the similar results in Section 3.1, we may thus regard equations (3.29) or (3.30) as our basic system of linear perturbation equations for the determination of s'. Clearly, in general, the objects H^I_p and $H^I_{(q)}$ will depend upon the choice of derivative operator ∇ on X. Nevertheless, it follows immediately from our constructions above that the expressions appearing on the left-hand side of equations (3.29) and (3.30) are independent of this choice.

Now, to analyze these matters still further and to make even a more suggestive link with equations (3.16), we give an additional word concerning $H_p^I(x)$ and $H_{(q)}^I(x)$. Let V and W be two vector spaces. The set of linear functions of V into W forms of course a vector space, which we denote by $L(V, W)$. It then follows from equations (3.27) and (3.28) that $H_p^I(x)$ is an element of $L(S_{p,x}, V_x^I)$, and $H_{(q)}^I(x)$ is an element of $L(S_{(q),x}, V_x^I)$. When for each $x \in X$ the vector space V_x^I can be identified with R, by virtue of $L(V, \mathrm{R}) = V^*$, where $V = S_{p,x}$ or $V = S_{(q),x}$, we reduce immediately to the situation in which $H_p^I(x) \in S_{p,x}^*$ and $H_{(q)}^I(x) \in S_{(q),x}^*$, so that equations (3.29) and (3.30) do not differ from those presented before in Section 3.1. Thus, we may conclude that equations (3.16) are indeed special cases of equations (3.29) and (3.30).

4. ANALYSIS OF THE GAUGE PROBLEM

4.1. Equivalence Classes of Perturbations

Because of the condition (2.24), there is a gauge freedom in covariant field theories corresponding to the group of diffeomorphisms of "space-time" X. To be completely explicit, the situation is simply this. Let σ: $X \to X$ be a diffeomorphism, and denote by $\sigma * s^A$ the image of s^A under σ. The definition of $\sigma * s^A$ is conventional[5] and appears, e.g., in Choquet-Bruhat *et al.* (1989). Since $\sigma * s^A$ is a tensor field on X of the same type as s^A, we can think of

$$\sigma * s := \bigoplus_{A=1}^{n} \sigma * s^A \tag{4.1}$$

as being the cross section of S. It then follows immediately from equations (2.23) and (2.24) that two different cross sections $s_{(1)}$ and $s_{(2)}$ of S are "physically" equivalent if there is a diffeomorphism σ: $X \to X$ which takes $s_{(1)}$ into $s_{(2)}$ [$\sigma * s_{(1)} = s_{(2)}$], and clearly $s_{(1)}$ satisfies the nonlinear field equations (2.23) if and only if $s_{(2)}$ does. Thus the solutions s of equations (2.23) can be unique only up to a diffeomorphism. Within the framework of a linear approximation, this implies that two perturbations $s'_{(1)}$ and $s'_{(2)}$ satisfying equations (3.30) represent the same perturbation of s_b if (and only if) they differ by the action of an "infinitesimal diffeomorphism" (Wald, 1984) on the background solution s_b of equations (2.23). An infinitesimal diffeo-

[5] Suppose that s^A is a vector field on X. Then we can define $\sigma * s^A$ by the relation $(\sigma * s^A)(x) := (\sigma^{-1})'(s^A(\sigma(x)))$ in which $(\sigma^{-1})'$ stands for the differential of the inverse mapping σ^{-1}: $X \to X$ at $\sigma(x)$. If s^A is a 1-form on X, we are justified in saying that $\sigma * s^A$ is the reciprocal image (pullback) of s^A under a differentiable mapping σ ($\sigma * s^A := \sigma^* s^A$, in the standard notation). With the help of these concepts, the general definition of $\sigma * s^A$ (when s^A is an arbitrary tensor field on X) follows readily.

morphism and its action on s_b are most conveniently described in terms of a vector field v on X. More precisely, using one-parameter groups of diffeomorphisms σ_ϵ, $\epsilon \in \mathcal{U}$, of X [these diffeomorphisms reduce to the identity as $\epsilon \to 0$ ($\sigma_0 = 1$)] and one-parameter families of exact solutions $s(\epsilon, \cdot)$ of equations (2.23), we can construct "new" one-parameter families of exact solutions $\bar{s}(\epsilon, \cdot) := \sigma_\epsilon * s(\epsilon, \cdot)$ obeying the condition $\bar{s}(0, \cdot) = s_b(\cdot) = s(0, \cdot)$ [see equation (3.2) for the definition of s_b] and hence verify that the change in a perturbation induced by v is the *Lie derivative* of s_b with respect to v:

$$\mathcal{L}_v s_b := \bigoplus_{A=1}^{n} \mathcal{L}_v s_b^A \tag{4.2}$$

Thus s' and $s' + \mathcal{L}_v s_b$ represent the same perturbation of s_b, and clearly s' satisfies the linear field equations (3.30) if and only if $s' + \mathcal{L}_v s_b$ does.

The set consisting of $\mathcal{L}_v s_b$ for all[6] vector fields v on X is written Γ_L; this set carries a natural structure of a vector space. For essentially obvious reasons, Γ_L is a subspace of the space Γ_C *whose elements are classical solutions of equations (3.30)*. Note that Γ_C is a proper subspace of Γ, the space of cross sections of S. The situation may therefore be summarized as follows. The object of most physical interest is not just one perturbation $s' \in \Gamma_C$, but a whole equivalence class of all perturbations $\bar{s}' \in \Gamma_C$ which are equivalent to s': two infinitesimal perturbations $s' \in \Gamma_C$ and $\bar{s}' \in \Gamma_C$ will be taken to be equivalent if there is a vector field v on X such that $\bar{s}' = s' + \mathcal{L}_v s_b$. The equivalence class of $s' \in \Gamma_C$ is denoted $[s']$ and is called the *gauge-invariant perturbation* associated with s'. In this way, we verify that the gauge-invariant perturbations are elements of Γ_C/Γ_L, the *quotient space* of Γ_C by Γ_L. The essential point in the theory of gauge-invariant perturbations is to describe the elements of this quotient space explicitly. These issues will be considered in a companion paper (Banach and Piekarski, 1997).

Another route to discussing the gauge problem is to introduce the equivalence class $[s']$ of $s' \in \Gamma$: two cross sections $x \mapsto s'(x)$ and $x \mapsto \bar{s}'(x)$ of S ($s' \in \Gamma$ and $\bar{s}' \in \Gamma$), *not necessarily satisfying equations (3.30), but still called the infinitesimal perturbations of* s_b (for reasons explained in Section 3.1), are equivalent if $\bar{s}'$ equals $s' + \mathcal{L}_v s_b$ for some vector field v on X of class C^k (k sufficiently large), (see footnote 6). Then we have the gauge-invariant perturbation $[s']$ associated with $s' \in \Gamma$ and the quotient space Γ/Γ_L which consists of $[s']$ for all $s' \in \Gamma$. Inspection shows that Γ_C/Γ_L is a *proper* subspace of Γ/Γ_L. We introduce here the quotient spaces Γ_C/Γ_L and

[6] Precisely speaking, v must be of class C^k (k sufficiently large); otherwise $\mathcal{L}_v s_b$ cannot be a classical solution of equations (3.30).

Γ/Γ_L, because the theory based on Γ_C/Γ_L only seems to be somewhat less convenient than that based on Γ_C/Γ_L and Γ/Γ_L.

4.2. Definition of Scalar Gauge-Invariant Variables

The steps presented so far in our analysis do not tell us directly how to use the equivalence classes of perturbations in practical calculations or whether such calculations are possible at all. As a matter of fact, in order to get at these issues, one will require a deeper understanding of the notion of a *gauge-invariant variable*, and, to facilitate this deeper understanding, it will be useful to first arrive at the notion of a scalar gauge-invariant variable (Section 4.2). This will, among many other things, serve to assemble the geometric machinary necessary for a general treatment of gauge-invariant variables (Section 4.3). In a companion paper (Banach and Piekarski, 1997) we shall indicate the mathematical conditions under which the equivalence classes of perturbations can be described in terms of a *finite* set of gauge-invariant variables.

With the foregoing considerations to guide us, we now turn our attention to scalar gauge-invariant variables. The first observation is simply this. In Section 2.1 we have defined, for *each* integer $r \geq 0$, the vector bundles $S_{(r)}$ and $S^*_{(r)}$ over X. One possible specification of r follows by recognizing that the nonlinear field equations (2.21) contain only the covariant derivatives[7] of $s \in \Gamma$ up to order q. This explains why in Section 3 we identified r with q. However, for the purpose of introducing the notion of a gauge-invariant variable, no such identifications are natural and all that can be said about r is that r is an integer ≥ 0 ($r \leq q$ or $r > q$).

With the usual convention that $D^r(\mathcal{L}_v s_b)$ is given by

$$D^r(\mathcal{L}_v s_b) := \bigoplus_{p=0}^{r} \nabla^p(\mathcal{L}_v s_b) \tag{4.3}$$

and that $(D^r(\mathcal{L}_v s_b))_x$ denotes the value of $D^r(\mathcal{L}_v s_b)$ at $x \in X$, we now define for each $x \in X$ the subspace $W_{(r),x}$ of $S_{(r),x}$ as follows: $s_{(r),x}$ belongs to $W_{(r),x}$ if and only if there exists a vector field v on X such that $s_{(r),x}$ equals $(D^r(\mathcal{L}_v s_b))_x$; hence

$$W_{(r),x} := \left\{ s_{(r),x} \in S_{(r),x}\colon \bigvee_{v} s_{(r),x} = (D^r(\mathcal{L}_v s_b))_x \right\} \tag{4.4}$$

Let $W_{(r)}$ be the disjoint union of the vector spaces $W_{(r),x}$ as x runs through X. By reason of this statement, $W_{(r)}$ is a subbundle of $S_{(r)}$. Suppose that $\langle \cdot, \cdot \rangle$ is

[7] In the remainder of this paper we assume that Γ (and hence $\Gamma_C \subset \Gamma$) consists of only such cross sections of S which are k times continuously differentiable (k sufficiently large).

a natural pairing of $S^*_{(r),x}$ and $S_{(r),x}$ into R (Bishop and Goldberg, 1968, p. 77). Then the subspace $F_{(r),x}$ of $S^*_{(r),x}$ can be defined as the set of linear forms $f_{(r),x} \in S^*_{(r),x}$ such that $\langle f_{(r),x}, s_{(r),x}\rangle = 0$ for all $s_{(r),x} \in W_{(r),x}$; we call this subspace the *annihilator* of $W_{(r),x}$ in $S^*_{(r),x}$. It is immediately verified that the disjoint union of the vector spaces $F_{(r),x}$ as x runs through X, denoted $F_{(r)}$, is a subbundle of $S^*_{(r)}$.

Let $\Gamma^*_{(r),F}$ be a vector space of cross sections of $F_{(r)}$. Note that, in our construction below, these cross sections are not necessarily continuous. For each $(x, f_{(r)}) \in X \times \Gamma^*_{(r),F}$ and each $[s'] \in \Gamma/\Gamma_L$, put (see footnote 7)

$$G(x, f_{(r)}, [s']) := \langle f_{(r)}(x), (D^r s')_x\rangle \tag{4.5}$$

where

$$(D^r s')_x := \bigoplus_{p=0}^{r} (\nabla^p s')_x \in S_{(r),x} \tag{4.6}$$

It is evident that in order to define $G(x, f_{(r)}, [s'])$, we have used one representative member of $[s'] \in \Gamma/\Gamma_L$, namely, the infinitesimal perturbation $s' \in \Gamma$ characterized by equation (3.5). Clearly, in view of the interpretation of s' as an element of Γ rather than Γ_C, we do not restrict our considerations above and below to the case when s' is a solution of equations (3.30). Since always

$$\langle f_{(r)}(x), (D^r s')_x\rangle = \langle f_{(r)}(x), (D^r(s' + \mathcal{L}_v s_b))_x\rangle \tag{4.7}$$

when $f_{(r)} \in \Gamma^*_{(r),F}$, the value of $\langle f_{(r)}, D^r \bar{s}'\rangle$ at $x \in X$ is completely independent of the choice of $\bar{s}' \in [s']$ and the object $G(x, f_{(r)}, \cdot)$ indeed defines a real-valued function on the quotient space Γ/Γ_L:

$$\Gamma/\Gamma_L \ni [s'] \mapsto G(x, f_{(r)}, [s']) \in \mathbf{R} \tag{4.8}$$

However, on account of equation (4.6), we easily verify that the dependence of $G(x, f_{(r)}, [s'])$ on $[s'] \in \Gamma/\Gamma_L$ is local in the following sense: the cross section $x \mapsto s'(x)$ is allowed to enter the definition (4.5) of $G(x, f_{(r)}, [s'])$ only through $s'(x)$ and the "space-time" covariant derivatives of s' up to order r evaluated at a point $x \in X$. In other words, the object $G(x, f_{(r)}, [s']) = \langle f_{(r)}(x), (D^r s')_x\rangle$ is such a linear and local algebro-partial differential consequence of an arbitrary C^r cross section s' of S that, for each vector field v on X of class C^{r+1}, the Lie derivative $\mathcal{L}_v s_b$ of s_b with respect to v can be added to s' without the need of replacing $\langle f_{(r)}(x), (D^r s')_x\rangle$ by the expression on the right-hand side of equation (4.7). Proceeding further, for each $f_{(r)} \in \Gamma^*_{(r),F}$ and each $[s'] \in \Gamma/\Gamma_L$ it will be natural to introduce the function $x \mapsto G(x, f_{(r)}, [s'])$, which is a mapping of X into R; *we call this mapping the scalar gauge-invariant variable of order r.* From these definitions it is plain that if only the vector bundle $F_{(r)}$ does exist, then there are infinitely many cross sections of $F_{(r)}$ and thus there are also infinitely many scalar gauge-invariant variables of order r.

Motivated by the above considerations, we now ask for what class of covariant field theories a gauge-invariant analysis of infinitesimal perturbations may be possible. First, let us observe that since $F_{(r),x}$ is in essence the *orthogonal complement* of $W_{(r),x}$ (Greub, 1975, p. 67), the dimensions of $F_{(r),x}$ and $W_{(r),x}$ are related to the dimension of $S_{(r),x}$ by

$$\dim (F_{(r),x}) + \dim (W_{(r),x}) = \dim (S_{(r),x}) \tag{4.9}$$

As usual, given the definition (4.4) of $W_{(r),x}$, it will be assumed that the background solution $s_b \in \Gamma$ of equations (2.21) is such that $\dim(W_{(r),x})$ and $\dim(F_{(r),x})$ [and hence $\dim(S_{(r),x})$] have values independent of the choice of $x \in X$. The next stage in the analysis is to see how the maximal possible dimension of $W_{(r),x}$ is influenced by changes in the dimension of X. According to equation (4.3), for any $s_b \in \Gamma$ the value of $D^r(\mathscr{L}_v s_b)$ at a point $x \in X$ depends not only on the value of the vector field v at the point $x \in X$, but also on its values in a neighborhood of $x \in X$—more precisely, $(D^r(\mathscr{L}_v s_b))_x$ is specified by giving $v(x)$ and $(\nabla^p v)_x$ up to order $r + 1$. Combining this fact with the definition of $W_{(r),x}$, we then find that

$$\dim (W_{(r),x}) \leq \sum_{p=1}^{r+2} N^p = \frac{N(1 - N^{r+2})}{1 - N} \tag{4.10}$$

where

$$N := \dim X \tag{4.11}$$

As regards the dimension of $S_{(r),x}$, the considerations of Section 2.1 allow us to write, for each $x \in X$,

$$\begin{aligned} \dim (S_{(r),x}) &= \sum_{p=0}^{r} \sum_{A=1}^{n} N^{R_A + r_A + p} \\ &= \frac{1 - N^{r+1}}{1 - N} \sum_{A=1}^{n} N^{R_A + r_A} \end{aligned} \tag{4.12}$$

Hence a sufficient condition that $\dim(F_{(r),x}) > 0$ can be stated as follows:

$$|1 - N^{r+2}| < |1 - N^{r+1}| \sum_{A=1}^{n} N^{R_A + r_A - 1} \tag{4.13}$$

Now, since n is a total number of tensor fields s^A in s [see equation (2.1)], the above inequality will be satisfied if, e.g.,

$$n > \frac{N\,|1 - N^{r+2}|}{|1 - N^{r+1}|} \tag{4.14}$$

Situations are known in which linearized theory gives an incorrect description of the collection of solutions of the field equations near a fixed background solution (Fischer *et al.*, 1980). However, if this approach is sufficient to capture the dominant effects of the nonlinear theory, as is usually the case (D'Eath, 1976; Banach and Makaruk, 1995), the inequality (4.14) has the unexpected corollary that a gauge-invariant treatment of infinitesimal perturbations will "almost always" be possible. This observation rigorously results from our analysis because, contrary to some of the opinions presented in the literature in recent years, the existence of $F_{(r)}$ and hence of $x \mapsto G(x, f_{(r)}, [s'])$ depends only slightly on the properties of the background. Thus, in all probability, nondegenerate linear perturbation theory for which sufficiently many gauge-invariant variables do exist that the equivalence classes of perturbations are completely determined by them (at least in principle) is a theory with sufficiently many tensor fields s^A on X (Banach and Piekarski, 1997).

As another consequence of our geometric approach, we are also able to arrive at the following result (Banach and Piekarski, 1996a): First consider metric theories of gravity (e.g., Einstein's gravity theory) and subsequently apply these theories to the construction of an almost-Robertson–Walker universe containing a perfect fluid (Ellis and Bruni, 1989). If so, the background metric g_b and the background fluid four-velocity u_b can be used to *naturally* make $S_{(r),x}$ into an inner product space, with the scalar product which we denote by $(\cdot, \cdot)$. Based on this scalar product, we immediately show that there is a linear isomorphism τ: $S_{(r),x} \to S^*_{(r),x}$ such that if $a_{(r),x}$ and $b_{(r),x}$ are arbitrary elements of $S_{(r),x}$, then $\langle \tau(a_{(r),x}), b_{(r),x}\rangle$ equals $(a_{(r),x}, b_{(r),x})$. Under these circumstances, it may be convenient not to distinguish between the annihilator $F_{(r),x}$ and its image $\tau^{-1}(F_{(r),x})$ in $S_{(r),x}$, but to regard them as the *same* vector space. This is called identification, and while it is not possible if there are no "prior geometric" elements such as g_b and u_b, in many cases of physical interest it leads to a great deal of economy of formulas and an alternative definition of gauge-invariant variables (Banach and Piekarski, 1996a, Section 4.1). Of course, we shall only identify spaces whenever we can introduce a *canonical* scalar product in $S_{(r),x}$, and the most general treatment of gauge-invariant variables is completely independent of the existence of this additional structure on X.

4.3. Definition of Tensorial Gauge-Invariant Variables

In Section 4.2 we defined the gauge-invariant objects G in such a way that for each choice of $(f_{(r)}, [s']) \in \Gamma^*_{(r),F} \times \Gamma/\Gamma_L$, the mapping $x \mapsto G(x, f_{(r)}, [s'])$ is a cross section of the vector bundle

$$\mathcal{R} := \bigcup_{x \in X} \mathrm{R}_x \tag{4.15}$$

where all the fibers R_x can be identified with R, the set of real numbers. However, instead of considering the vector bundle $\mathcal{R}$, we may more generally consider the vector bundle

$$T := \bigcup_{x \in X} T_x \tag{4.16}$$

whose cross sections are tensor fields on X (Dieudonné, 1972; Choquet-Bruhat *et al.*, 1989). Consequently, the primary task now is to generalize to the context of T the notion of a scalar gauge-invariant variable of order r.

To this end, let $S^{\star}_{(r),x} := L(S_{(r),x}, T_x)$ be the set of linear functions of $S_{(r),x}$ into T_x, and denote by $S^{\star}_{(r)}$ the disjoint union of the $S^{\star}_{(r),x}$ as x runs through X:

$$S^{\star}_{(r)} := \bigcup_{x \in X} S^{\star}_{(r),x} \tag{4.17}$$

From these definitions it follows immediately that since $S^{\star}_{(r),x}$ forms for each $x \in X$ a vector space, we may regard $S^{\star}_{(r)}$ as being the vector bundle over X. By analogy with what was said earlier in connection with equations (3.30), given $s_{(r),x} \in S_{(r),x}$, an element $a^{\star}_{(r),x}(s_{(r),x}) \in T_x$ of the range set of $a^{\star}_{(r),x} \in S^{\star}_{(r),x}$ is called a value of $a^{\star}_{(r),x}$ on $s_{(r),x}$ (or the image of $s_{(r),x}$ under $a^{\star}_{(r),x}$) and is denoted $\langle a^{\star}_{(r),x}, s_{(r),x} \rangle$. Of course, if $\{e_K(x)\}$ is a basis of T_x, and if we put

$$a^{\star}_{(r),x} = \sum_K a^{\star(K)}_{(r),x}\, e_K(x) \tag{4.18}$$

where $a^{\star(K)}_{(r),x}$ are linear functions on $S_{(r),x}$ with values in R, then we can explicitly characterize the action $a^{\star}_{(r),x}$ on $s_{(r),x}$ by

$$\langle a^{\star}_{(r),x}, s_{(r),x} \rangle = \sum_K \langle a^{\star(K)}_{(r),x}, s_{(r),x} \rangle\, e_K(x) \tag{4.19}$$

Now suppose that the subspace $W_{(r),x}$ of $S_{(r),x}$ has exactly the same meaning as in Section 4.2, and describe the *annihilator* $\overline{F}_{(r),x}$ of $W_{(r),x}$ in $S^{\star}_{(r),x}$ by saying that $f_{(r),x}$ belongs to $\overline{F}_{(r),x} \subset S^{\star}_{(r),x}$ if and only if $\langle f_{(r),x}, s_{(r),x} \rangle = 0$ for all $s_{(r),x} \in W_{(r),x}$. It will be convenient to set

$$\overline{F}_{(r)} := \bigcup_{x \in X} \overline{F}_{(r),x} \tag{4.20}$$

Obviously, this definition gives an interpretation of $\overline{F}_{(r)}$ as a (vector) subbundle of $S^{\star}_{(r)}$. Likewise, for each integer $r \geq 0$, the (vector) subbundle $W_{(r)}$ of $S_{(r)}$ can be defined:

$$W_{(r)} := \bigcup_{x \in X} W_{(r),x} \tag{4.21}$$

Once we have introduced $\overline{F}_{(r)}$ via equation (4.20), the remainder of the specification of the gauge-invariant construction is straightforward. First, it is not difficult to see that all the considerations and all the results in Section 4.2 can be transposed to the context of T simply by replacing $(\mathcal{R}, S^{\star}_{(r)}, W_{(r)}, F_{(r)})$ throughout by $(T, S^{\star}_{(r)}, W_{(r)}, \overline{F}_{(r)})$. To illustrate this, let us remark the following: if $\Gamma^{\star}_{(r),F}$ is a vector space of cross sections of $\overline{F}_{(r)}$, then for each $(x, f_{(r)}) \in X \times \Gamma^{\star}_{(r),F}$ and each $[s'] \in \Gamma/\Gamma_L$ it will be possible to define the gauge-invariant object $G(x, f_{(r)}, [s']) \in T_x$ by

$$G(x, f_{(r)}, [s']) := \langle f_{(r)}(x), (D^r s')_x \rangle \tag{4.22}$$

Again we verify that since

$$\langle f_{(r)}(x), (D^r s')_x \rangle = \langle f_{(r)}(x), (D^r(s' + \mathcal{L}_v s_b))_x \rangle \tag{4.23}$$

the value of $\langle f_{(r)}, D^r \bar{s}' \rangle$ at $x \in X$ is completely independent of the choice of $\bar{s}' \in [s']$ and thus the mapping $[s'] \mapsto G(x, f_{(r)}, [s'])$ may be regarded as a linear function of Γ/Γ_L into T_x. The analogy to equation (4.8) is immediate, and further discussion proceeds in the same way as in Section 4.2. More precisely, the next stage in the construction is to observe that, after freely choosing $f_{(r)} \in \Gamma^{\star}_{(r),F}$ and $[s'] \in \Gamma/\Gamma_L$, we obtain the mapping $x \mapsto G(x, f_{(r)}, [s'])$, which is a cross section of T. *Here and henceforth, this cross section will be called the gauge-invariant variable of order r.* As we shall demonstrate in our analysis in a companion paper (Banach and Piekarski, 1997), such a generalization of the notion of a scalar gauge-invariant variable is the most convenient one for working less abstractly with the elements $[s']$ of a quotient space Γ/Γ_L, i.e., for explicitly specifying the conditions under which the equivalence classes of perturbations are uniquely determined from a knowledge of only finitely many gauge-invariant variables. Note also that although some aspects of the definitions of $G(x, f_{(r)}, [s'])$ depend upon the choice of derivative operator ∇ on X, nevertheless the notion of a gauge-invariant variable of order r and the property (4.23) do not by virtue of our remarks at the beginning of Section 2.2.

The above discussion leads to the following overall picture: any cross section of $\overline{F}_{(r)}$, *not necessarily continuous*, can in a sense be identified with the gauge-invariant variable of order r, and if only the vector bundle $\overline{F}_{(r)}$ does exist, as is quite often the case, then there are infinitely many cross sections of $\overline{F}_{(r)}$ and thus there are also infinitely many gauge-invariant variables of order r. With regard to the choice of an integer $r \geq 0$ and a tensor bundle T in the definition of $\overline{F}_{(r)}$, this choice depends mostly on us, and different *possible* choices of r and T give rise to different gauge-invariant variables.

What is the relationship between the notion of a tensorial gauge-invariant variable defined above and the notion of a scalar gauge-invariant variable arising from the construction of Section 4.2? The tensor field $G(\cdot, f_{(r)}, [s'])$

on X determined by equation (4.22) has of course an expression in terms of scalar gauge-invariant variables. To see this, we first introduce the frame $\{e_K\}$ of T over X and subsequently decompose $f_{(r)} \in \overline{F}_{(r)}$ as

$$f_{(r)} = \sum_K f_{(r)}^K e_K \tag{4.24}$$

where the objects $f_{(r)}^K$ are cross sections of the vector bundle $F_{(r)}$ defined in Section 4.2. Naturally, substituting this decomposition for $f_{(r)}$ into equation (4.22) yields the result

$$G(x, f_{(r)}, [s']) = \sum_K G^K(x, f_{(r)}, [s'])e_K(x) \tag{4.25}$$

in which

$$G^K(x, f_{(r)}, [s']) := \langle f_{(r)}^K(x), (D^r s')_x \rangle \tag{4.26}$$

Since the cross sections $x \mapsto G^K(x, f_{(r)}, [s'])$ of $\mathfrak{R}$ [see equation (4.15)] serve to define scalar gauge-invariant variables of order r, it is obvious from the decomposition (4.25) for $G(x, f_{(r)}, [s'])$ that the original gauge-invariant variable, namely the mapping $x \mapsto G(x, f_{(r)}, [s'])$, can be expressed in terms of these cross sections. Of course, the replacement of G by $\{G^K\}$ is not a fully covariant activity, as it depends on the particular frame $\{e_K\}$ chosen. Nevertheless, such an approach helps us considerably simplify the discussion of a gauge problem, provided we drop the requirement that our analysis be fully covariant. Moreover, in many cases of physical interest (Banach and Piekarski, 1996a–d) the properties of a background solution s_b to equations (2.23) dictate the use of preferred frames $\{e_K\}$.

5. THE GEOMETRIC SIGNIFICANCE OF $H_{(q)}^I$

As noted already in Section 2.2, the relation (2.24) for H^I, $I = 1,2, \ldots, m$, establishes a natural prescription for constructing diffeomorphism-invariant, covariant field theories. Because of this relation, in the case of interest to us—namely the partial differential equations governing linearized perturbations—the essential two properties of the expression $\langle H_{(q)}^I, D^q s' \rangle$ appearing on the left-hand side of equation (3.30) are that it (i) is linear in $s' \in \Gamma$ (see footnote 7) and (ii) satisfies for each C^{q+1} vector field v on X a condition of the form

$$\langle H_{(q)}^I, D^q s' \rangle = \langle H_{(q)}^I, D^q(s' + \mathcal{L}_v s_b) \rangle \tag{5.1}$$

Note that, in interpreting this condition, there is no need to assume that $s' \in \Gamma$ is a solution of the linearized field equations. We then may view the mapping $x \mapsto \langle H_{(q)}^I(x), (D^q s')_x \rangle$, which is a cross section of the vector bundle

V^l (see Section 3.2 for the definition of V^l), as a gauge-invariant variable of order q.

The basic geometrical content of this statement should be more or less obvious. After identifying T with V^l in the general construction of Section 4.3, we can think of the mapping $x \mapsto H^l_{(q)}(x)$ as being the cross section of $\overline{F}_{(q)}$. Since this interpretation of $H^l_{(q)}$ is rather important and seems necessary if one wishes to find a manifestly gauge-invariant form of equations (3.30), it will be developed further in a companion paper (Banach and Piekarski, 1997).

6. DISCUSSION AND CONCLUDING REMARKS

The main goal of this paper was to provide a coherent, self-contained introduction to the geometric formulation of linear perturbation theory for covariant field equations. Because of the condition (2.24), this formulation differs from the "conventional" one (Vinogradov *et al.*, 1986) in that the solutions of the linearized field equations can be unique only up to an "infinitesimal diffeomorphism" of the "space-time" manifold X, i.e., two perturbations s' and $\bar{s}'$ satisfying equations (3.30) characterize the same perturbation of the "background" solution s_b if (and only if) there exists a vector field v on X such that $\bar{s}' - s'$ is the *Lie derivative* $\mathcal{L}_v s_b$ of s_b with respect to v. Consequently, to have genuine physical significance, gauge-invariant variables should be constructed from the objects canonically present in the problem, here s_b and s', without reliance on artificially introduced concepts, such as the notion of a point identification map between two different manifolds X and $\overline{X}$ of independent variables.

In Section 4, we have presented the construction of gauge-invariant variables without ever specifying the detailed form of covariant field equations and without ever analyzing the symmetry properties of the background, if any. This contrasts sharply with most presentations of linear perturbation theory, wherein one decides at the outset to define H^l and ends up constructing gauge-invariant variables for particular choices of s_b. Thus, in our discussion of the gauge problem, it is important that we distinguish clearly between the essential input in the theory (namely, the general definition of the annihilator $\overline{F}_{(r),x}$ of $W_{(r),x}$ in $S^\star_{(r),x}$) and inessential input (namely, the choice of a concrete form of H^l and s_b). Some of the results obtained in this paper are based on earlier published works devoted to applications of the linear approximation method to the analysis of Einstein's gravity theory for the description of an almost-Robertson–Walker universe (Banach and Piekarski, 1994a–c, 1996a–d). These results were considerably generalized here and presented in a form completely independent of how the objects H^l and s_b are chosen, with more discussion on the motivation and explanation for the geometrical aspects of the theory than space would allow in "normal technical" investigations, and they were given in one place where there would then be a more unified and coherent explication of the subject.

Another interesting question has already been alluded to in the Introduction and in Section 4.3. Could a finite set of gauge-invariant variables be applied successfully to a unique characterization of the equivalence classes of perturbations? Or, more precisely, will a tractable analytical expression for $[s'] \in \Gamma/\Gamma_L$ exist? That these sorts of problems actually do arise has in fact been demonstrated explicitly in our previous papers (Banach and Piekarski, 1996a–d). As an illustration, beginning from Einstein's gravity theory, it was demonstrated there that, in the case of an almost-Robertson–Walker universe (Ellis and Bruni, 1989) dominated by a collisionless gas or a general perfect fluid, the complete characterization of cosmological perturbations can be obtained if one defines in a suitable way 17 or 18 "geometrically" independent, not identically vanishing gauge-invariant variables. One can think of these *basic* variables, denoted collectively by ω, as having at least three aspects. First, ω provides a mathematically simplest representation of the gauge-invariant perturbation $[s']$. In fact, $[s']$ is uniquely determined from ω and vice versa. Second, any gauge-invariant quantity is obtainable directly from the basic variables ω through purely local (i.e., algebraic and differential) operations. Third, a complete set of propagation equations can be derived that involves only ω. These equations are physically more transparent than the usual ones, because spurious "gauge mode" solutions are automatically excluded.

Given the viewpoint adopted above, in a companion paper (Banach and Piekarski, 1997) our attention will, in large part, focus upon such conceptual matters as how one might generalize the definition of ω so as to apply it to arbitrary diffeomorphism-invariant, covariant field theories. Thus, for example, one would like to specify the conditions under which a finite set of gauge-invariant variables, still denoted by ω, suffices to obtain an explicit description of the equivalence classes of perturbations. Similarly, one might like to show that any gauge-invariant quantity can be expressed locally in terms of ω. These sorts of problems will be discussed in a companion paper (Banach and Piekarski, 1997). A new nontrivial example of ω [as compared with those already presented (Banach and Piekarski, 1996a–d)] will also be pointed out there.

Despite these results and perspectives, a number of unresolved issues remain. Specifically, it seems very important to determine how, at least in principle, one would go beyond the "simple" linear approximation; and, as a corollary to this, one would clearly like to verify whether a covariant and gauge-invariant treatment of nonlinear perturbations can really be justified on a geometrical level. This problem arises in particular in examining the structure of perturbation theory for Einstein's field equations and their modifications such as the Brans–Dicke and Hellings–Nordtvedt equations (Brans and Dicke, 1961; Hellings and Nordtvedt, 1973).

APPENDIX. SOME AUXILIARY TECHNICAL DEFINITIONS AND CONCEPTS

In Section 3, we were naturally led to define the "derivative of H^I and H^I_K with respect to $\overline{w}^p \in S_{p,x}$ at $\overline{w}^p = w^p$" to be a linear form on $S_{p,x}$, i.e., an element of the vector space $S^*_{p,x}$ dual to $S_{p,x}$. This derivative, denoted in Section 3.1 by dH^I/dw^p and in Section 3.2 by dH^I_K/dw^p, is a special case of the more general concept, which can be described as follows.

Let W be a finite-dimensional vector space and suppose that Υ: $W \to \mathrm{R}$ is the mapping of class C^1 which associates with $w \in W$ the real number $\Upsilon(w)$. If $\{e^K\}$ is a basis of W, then every vector $w \in W$ is uniquely expressible in the form

$$w = \sum_K w_K e^K \tag{A.1}$$

where the objects w_K are components of w with respect to $\{e^K\}$. Next, let $\{e_K\}$ be a basis of W^* dual to the basis $\{e^K\}$ of W. We can then define the *derivative of* Υ *with respect to* w, denoted for brevity by $d\Upsilon/dw$, to be a linear form on W, i.e., an element of the vector space W^* dual to W. More precisely, after introducing the basis $\{e^K\}$ of W, it will be possible to regard $\Upsilon(w)$ as a differentiable function of the components w_K of w and thus characterize the aforementioned derivative by

$$\frac{d\Upsilon}{dw} := \sum_K e_K \frac{\partial \Upsilon}{\partial w_K} \tag{A.2}$$

Elementary inspection shows that the notion at which we arrive in this way does not depend on the particular basis $\{e^K\}$ of W chosen.

As special cases which illustrate this general construction, identify the vector space W with $S_{p,x}$ and the mapping Υ: $W \to \mathrm{R}$ with either

$$H^I(x, w^0, \ldots, w^{p-1}, \bullet, w^{p+1}, \ldots, w^q): \quad S_{p,x} \to \mathrm{R} \tag{A.3a}$$

or

$$H^I_K(x, w^0, \ldots, w^{p-1}, \bullet, w^{p+1}, \ldots, w^q): \quad S_{p,x} \to \mathrm{R} \tag{A.3b}$$

Then we obtain the objects dH^I/dw^p and dH^I_K/dw^p of Section 3 [see especially equations (3.11) and (3.21)].

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